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Gravity waves in a circular well

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The natural frequencies of gravity waves in a circular well that is bounded above by a free surface and below by a semi-infinite reservoir are approximated by neglecting the off-diagonal terms of the characteristic determinant (single-mode approximation) and invoking the known results for an aperture in a half-space (well of zero depth). A parallel argument yields the corresponding results for a two-dimensional well (a slot). Comparison with Molin's (2001) numerical results for a slot suggests that the error in the single-mode approximation is $\lesssim 1\%$.

1. Introduction

I consider here the small, gravitational oscillations of a liquid in a circular well of radius a and depth h that is bounded above by a free surface and below by a semi-infinite reservoir. The rectangular well has been considered by Molin (2001) as a model of a 'well-bay' or 'moonpool' in a ship. Molin also has considered a slot (two-dimensional) well. The limiting case, $h \to 0$, for circular and strip apertures has been treated extensively (see Henrici, Troesch & Wuytack 1970; Troesch & Troesch 1972; Miles 1972).

I attack the boundary-value problem in § 2 through a Fourier–Bessel series in the well and a Fourier–Bessel integral in the reservoir. The matching of these representations in the mouth yields an integral equation for the determination of the free oscillations and their frequencies. In § 3, I obtain approximations to these frequencies by neglecting the off-diagonal terms in the characteristic determinant. This permits a direct calculation of the eigenvalue for the Helmholtz mode and the calculation of the higher eigenvalues from those for $h = 0$ for either the circular well or, through a parallel argument, its two-dimensional counterpart (a slot). Comparison with Molin's (2001) numerical results for a slot suggests that the error in the present approximations is $\lesssim 1\%$.

2. The boundary-value problem

The boundary-value problem for the circular well is governed by

$$\nabla^2\phi = 0 \quad (0 < r < 1,\ 0 < z < h; \quad 0 < r < \infty,\ -\infty < z < 0), \tag{2.1}$$

$$\phi_z = w(r,\theta) \quad (z = 0), \quad \phi_z = \lambda\phi \quad (0 < r < 1,\ z = h), \tag{2.2a,b}$$

$$\phi_r = 0 \quad (r = 1,\ 0 < z < h), \tag{2.3}$$

and

$$\phi \to 0 \quad (z < 0,\ R \equiv (r^2 + z^2)^{1/2} \to \infty), \tag{2.4}$$

where $\phi = \phi(r, z, \theta)$ is the complex amplitude of the velocity potential, which we assume to be simple harmonic with angular frequency ω; r, θ, z are dimensionless cylindrical coordinates, with a, the radius of the well, as the unit of length; $w(r, \theta)$ is the complex amplitude of the vertical velocity in the mouth ($0 < r < 1,\ z = 0$); $w \equiv 0$ in $r > 1$;

$$\lambda \equiv \omega^2 a/g \tag{2.5}$$

is the eigenvalue.

Separating the azimuthal dependence through the Fourier expansions

$$\phi(r, \theta, z) = \sum_{m=0}^{\infty} \phi_m(r, z)\mathrm{e}^{\mathrm{i}m\theta} \quad (0 < z < h), \quad w(r, \theta) = \sum_{m=0}^{\infty} w_m(r)\mathrm{e}^{\mathrm{i}m\theta}, \tag{2.6a, b}$$

substituting (2.6a, b) into (2.1)–(2.3), expanding w_m in the Fourier–Bessel series

$$w_m(r) = \sum_n (W_n/I_n) J_m(k_n r), \quad W_n = \int_0^1 w_m(r) J_m(k_n r) r\, \mathrm{d}r, \tag{2.7a, b}$$

where the parametric dependence of k_n, I_n, W_n and, subsequently, Z_n and κ_n, on m is implicit,

$$I_n \equiv \int_0^1 J_m^2(k_n r) r\, \mathrm{d}r = \left(\frac{k_n^2 - m^2}{2k_n^2} \right) J_m^2(k_n), \tag{2.8}$$

and the summation is over the complete, orthogonal set determined by

$$J_m'(k_n) \quad (0 < k_1 < k_2 < \cdots) \tag{2.9}$$

($k_0 = 0$ is admissible if and only if $h > 0$), and solving (2.1) and (2.2a, b) through separation of variables, we obtain

$$\phi_m(r, z) = \sum_n (k_n I_n)^{-1} W_n J_m(k_n r) Z_n(z) \quad (0 < r < 1,\ 0 < z < h), \tag{2.10}$$

where

$$Z_n(z) = \sinh k_n z + \kappa_n \cosh k_n z, \quad \kappa_n = \frac{k_n - \lambda \tanh k_n h}{\lambda - k_n \tanh k_n h}. \tag{2.11a, b}$$

The solution of (2.1), (2.2a) and (2.4) in the reservoir is given by the Fourier–Bessel integral

$$\phi_m(r, z) = \int_0^1 w_m(\eta)\eta\, \mathrm{d}\eta \int_0^\infty J_m(kr) J_m(k\eta) \mathrm{e}^{kz}\, \mathrm{d}k \quad (z < 0). \tag{2.12}$$

Equating the representations (2.10) and (2.12) in the mouth, we obtain

$$\sum_n (\kappa_n/k_n I_n) W_n J_m(k_n r) = \int_0^1 w_m(\eta)\eta\, \mathrm{d}\eta \int_0^\infty J_m(kr) J_m(k\eta)\, \mathrm{d}k \quad (0 < r < 1) \tag{2.13}$$

for the determination of w_m. This integral equation may be transformed to a set of linear equations for the determination of the Fourier–Bessel coefficients W_n by substituting w_m from (2.7), multiplying by $rJ_m(k_l r)$, and integrating over $0 < r < 1$. The end result is

$$\sum_l C_{ln} W_n = 0, \quad C_{ln} = \delta_{ln}(\kappa_l/k_l) I_l - \int_0^\infty Q_l(k) Q_n(k)\, \mathrm{d}k, \tag{2.14a, b}$$

where

$$Q_n(k) = \int_0^1 J_m(k_n r) J_m(kr) r \, \mathrm{d}r = k(k_n^2 - k^2)^{-1} J_m(k_n) J_m'(k). \tag{2.15}$$

3. Single-mode approximation

The eigenvalues of the system (2.14) are determined by the determinantal condition $|C_{ln}| = 0$. The neglect of the off-diagonal terms in this condition or, equivalently, the retention of only a single term in the Fourier–Bessel expansion (2.7*a*) (Molin's 'single mode approximation') yields

$$\kappa_n = \frac{k_n}{I_n} \int_0^\infty Q_n^2(k) \, \mathrm{d}k = \left(\frac{2k_n^3}{k_n^2 - m^2} \right) \int_0^\infty \frac{J_m'^2(k) k^2 \, \mathrm{d}k}{(k^2 - k_n^2)^2}. \tag{3.1}$$

(The integrand is finite at $k = k_n$ by virtue of (2.9).)

The integral in (3.1) is intractable except for the Helmholtz mode ($m = n = 0$). Substituting κ_n from (2.11*b*), letting $m = 0$, dividing by k_n, and letting $k_n \to k_0 \to 0$, we obtain

$$\frac{\kappa_n}{k_n} \to \frac{1}{\lambda} - h = 2 \int_0^\infty \frac{J_1^2(k) \, \mathrm{d}k}{k^2} = \frac{8}{3\pi} \quad (m = n = 0), \tag{3.2}$$

from which it follows that

$$\lambda = \frac{1}{h + \alpha}, \quad \alpha = \frac{8}{3\pi} = 0.849. \tag{3.3a, b}$$

This result also follows by analogy with the acoustical problem of a plane wave in a cylindrical tube that terminates in a half-space (Rayleigh 1896, § 312).

Returning to (3.1), we remark that the right-hand side thereof is, and hence the left-hand side also must be, independent of h. Letting $h = 0$ and $\lambda = \lambda_n^0$ in (2.11*b*), we obtain

$$\kappa_n = k_n / \lambda_n^0 \quad (k_n > 0), \tag{3.4}$$

where λ_n^0 are the eigenvalues for the aperture problem (Miles 1972, table 1).† It then follows from (2.11*b*) that the eigenvalues for $h > 0$ are given by

$$\frac{\lambda_n^h}{k_n} = \frac{1 + \kappa_n \tanh k_n h}{\kappa_n + \tanh k_n h} = \coth (k_n h + \tanh^{-1} \kappa_n) \quad (k_n > 0). \tag{3.5}$$

We remark that (3.5) implies $\lambda_n^0 > \lambda_n^h > k_n$ for all $n \geqslant 1$.

Troesch & Troesch (1972) infer from their numerical results that

$$\lambda_n^0 \sim \pi \left(n + \tfrac{1}{2}|m - 1| - \tfrac{1}{8} \right) \quad (m \geqslant 0, k_n \to \infty), \tag{3.6}$$

which, together with the corresponding result for the zeros of (2.9).

$$k_n \sim \pi \left(n + \tfrac{1}{2}|m - 1| - \tfrac{1}{4} \right), \tag{3.7}$$

yields

$$\lambda_n^h \sim k_n + \tfrac{1}{8}\pi \exp(-2k_n h) \quad (k_n \to \infty). \tag{3.8}$$

† More precisely, λ_n^0 in (3.4) and (3.5) are the single-mode approximations to the eigenvalues for the aperture problem, but the error in using the true λ_n^0 is of the same order as that already implicit in the present single-mode approximation.

h	$\lambda_1^h(3.5)$	Molin (23)	Molin (numerical)
0	2.006	2.030	2.006
0.05	1.936	1.955	1.955
0.1	1.877	1.893	1.902
0.2	1.789	1.800	1.811
0.4	1.684	1.689	1.696
0.6	1.630	1.633	1.636
0.8	1.602	1.604	1.605
1.0	1.587	1.588	1.589

TABLE 1. The eigenvalues for a slot well for $n = 1$, as calculated from (3.5), Molin's (23), and Molin's numerical integration (last column).

4. Two-dimensional comparisons

The results (3.4), (3.5) and (3.8) also hold for a two-dimensional well of width $2a$, for which $k_n = \frac{1}{2}n\pi$ (including both the even and odd modes) and the λ_n^0 are given by table 3 in Miles (1972). The two-dimensional results for $n = 1$, as given by (3.5), Molin's (23), and Molin's numerical results (personal communication), are compared in table 1. The present approximation differs from Molin's (23) only in their approximations to κ_n, 0.783 and 0.774, respectively. The error in both approximations is second-order in the off-diagonal terms of the characteristic matrix. Molin's (23) is slightly more accurate than (3.5) except for rather small h, but it does not give the correct limit for $h \downarrow 0$; both approximations are within 1% of the numerical results. The asymptotic approximation (3.8) agrees to three decimals with (3.5) for $h \geqslant 0.4$ and with the numerical results for $h \geqslant 1$.

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